SOME REMARKS ON ANOTHER PROOF OF GEOMETRICAL PALEY–WIENER THEOREMS FOR THE DUNKL TRANSFORM

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ABSTRACT. We argue that another proof by Trimèche of the geometrical form of the Paley–Wiener theorems for the Dunkl transform is not correct.

1. INTRODUCTION

In [3] a proof of the geometrical form of the Paley–Wiener theorems for the Dunkl transform is presented. It is argued in [1] that this proof is not correct. A new proof has appeared in [4]. In this note we argue that this new proof is not correct.

The material which is presented below has previously been communicated to the author. It is our opinion that at this moment the geometrical forms of the Paley–Wiener theorems for the Dunkl transform are still unproven, and that only partial results have been established [2].

2. ARGUMENTS

In [4] the geometric form of the Paley-Wiener theorem for the Dunkl transform is stated for functions as Theorem 5.1. Our arguments concern the proof of this theorem. In formulating them, we will use the notation and definitions of [4].

In the proof of Theorem 5.1, the function ϕ on $\mathbb{R}^d$ is defined in equation (5.3) as

$$(2.1) \qquad \phi(x) = \int_{\mathbb{R}^d} f(y)K(iy,x)\frac{\omega_k(y)}{(1+\|y\|^2)^p}dy \quad (x \in \mathbb{R}^d).$$

Here f is a function of Paley–Wiener type corresponding to a W-invariant compact convex subset E of $\mathbb{R}^d$, as in the statement of Theorem 5.1, and p is an integer such that $p > \gamma + \frac{d}{2} + 1$. Since the constant γ is assumed to be strictly positive in line -10 of page 4, we see that that $p \geq 2$.

After a computation involving Riemann sums and contour integration, it is then concluded on line 9 of page 13 that ϕ has support in the set E. In particular, ϕ has compact support. Since it has already been observed in line 10 on page 11 that ϕ is smooth, the easy part of the Paley–Wiener theorem implies that the Dunkl transform of ϕ is of Paley–Wiener type. In particular, the Dunkl transform of ϕ has an entire extension to $\mathbb{C}^d$. The inversion theorem and our equation (2.1) therefore show that the function

$$(2.2) \qquad\qquad \frac{f(y)}{(1+\|y\|^2)^p}$$

on $\mathbb{R}^d$ has an entire extension to $\mathbb{C}^d$.

Now take E to be an arbitrary closed ball, centered at the origin. Certainly, if f is the ordinary Fourier transform of a smooth function with support in E, then f is of Paley–Wiener type corresponding to the W-invariant compact convex set E, so that the above reasoning applies to f.

Combining the previous two paragraphs, we conclude that the function in our equation (2.2) has an entire extension from $\mathbb{R}^d$ to $\mathbb{C}^d$, for all f which are the ordinary Fourier transform of a compactly supported smooth function. Since f has then itself an entire extension, we see that the meromorphic function

$$\frac{f(y)}{(1 + (y, y))^p}$$

on $\mathbb{C}^d$ has a removable singularity along the divisor $\{y \in \mathbb{C}^d \mid (y, y) = -1\}$, where $(\,.\,,\,.\,)$ is the standard bilinear form on $\mathbb{C}^d$. Since $p \geq 2 > 0$, we conclude that each function f, which is the entire extension of the ordinary Fourier transform of a smooth compactly supported function, vanishes on this divisor.

Now let f be the Fourier transform of a non-zero positive smooth compactly supported function. Then $f(i, 0, \ldots, 0) > 0$. Since $((i, 0, \ldots,), (i, 0, \ldots,)) = -1$ we must also have $f(i, 0, \ldots,) = 0$ by the reasoning above. This is a contradiction.

The above arguments are not based on the details of the computation with Riemann sums and contour integration, but they are concerned with the impossibility of the support of ϕ being contained in E. These arguments therefore show not only that this computation contains a technical inaccuracy, but they also show that the technique of this computation can not be corrected, since the conclusion which the computation is supposed to yield does not hold.

References

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[3] K. Trimèche, *Paley-Wiener theorems for the Dunkl transform and Dunkl translation operators*, Integral Transforms Spec. Funct. **13** (2002), 17–38.

[4] K. Trimèche, *Another proofs of the geometrical forms of Paley–Wiener theorems for the Dunkl transform and inversion formulas for the Dunkl interwining operator and for its dual*, Preprint (2004). ArXiv: math.CA/0405050 v1.

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